

MAS234

UNIVERSITY OF NEWCASTLE UPON TYNE

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 2 2002/2003
-------------------------

MAS234

Applied Probability

Time allowed: 1 hour 30 minutes

*Credit will be given for ALL answers to questions in Section A, and for the best TWO answers to questions in Section B. No credit will be given for other answers and students are strongly advised not to spend time producing answers for which they will receive no credit.*

*Marks allocated to each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.*

*There are THREE questions in Section A and THREE questions in Section B.*

## SECTION A

A1. Suppose that a random variable  $X$  has a Binomial distribution with parameters  $n$  and  $p$ . Find  $E \left[ (1/3)^X \right]$ .

[12 marks]

A2. In a particular type of random walk process, a particle starts from 0 at time 0, and then, at each time  $1, 2, 3, \dots$ , either moves one step to the right with probability  $p$  or stays at the same position with probability  $q = 1 - p$ . If we denote by  $S_n$  the distance to the right which the particle has moved from 0 after  $n$  steps, then  $S_0 = 0$  and, assuming that the moves are independent of each other,

$$S_n = \sum_{j=1}^n X_j \quad n = 1, 2, \dots$$

where  $X_1, X_2, \dots$  are independent, identically distributed random variables such that

$$P(X_j = 1) = p \quad \text{and} \quad P(X_j = 0) = q = 1 - p.$$

- (i) Given  $n = 1, 2, \dots$ , what is the distribution of  $S_n$ ?
- (ii) For  $n = 1, 2, \dots$ , what is the expected position of the particle at time  $n$ ?

[12 marks]

A3. Consider a random variable  $X$  with probability function

$$p_x = P(X = x), \quad x = 0, 1, 2, \dots,$$

and probability generating function  $G_X(s)$ .

- (i) Define the random variable  $Y = aX + b$ , where  $a$  and  $b$  are positive integers. If  $G_Y(s)$  is the probability generating function of  $Y$ , show that

$$G_Y(s) = s^b G_X(s^a).$$

- (ii) If  $p_x = \frac{1}{2^{x+1}}$ ,  $x = 0, 1, 2, \dots$ , determine  $G_X(s)$  and  $G_Y(s)$ . Use  $G_Y(s)$  to show that the mean of  $Y$  is  $a + b$ .

[16 marks]

## SECTION B

B4. Let  $X_n$ ,  $n = 0, 1, 2, \dots$  denote the number in the  $n$ -th generation of a branching process with  $X_0 = 1$ , and suppose that  $X_n$  has probability generating function (p.g.f.) which is denoted by  $G_n(s)$ . Let  $Z_\alpha$  be the number of offspring of individual  $\alpha$ , and write  $E[Z_\alpha] = \mu$  and  $Var(Z_\alpha) = \sigma^2$ .

(a) Use a conditioning argument to prove that

$$E[X_2] = \mu^2.$$

(b) Consider a branching process with:

$$Z_\alpha = \begin{cases} 0, & \text{with probability } \frac{1}{4} \\ 1, & \text{with probability } \frac{1}{8} \\ 2, & \text{with probability } \frac{5}{8}. \end{cases}$$

For questions (i)-(iv) below, you may use the following result without proof:

$$G_n(s) = G_{n-1} \{G_1(s)\}.$$

- (i) Obtain  $G_1(s)$  and  $G_2(s)$  for this process..
- (ii) From (i) deduce the probability function of  $X_2$ , and confirm that  $E[X_2] = \mu^2$  for this process.
- (iii) Find formulae for  $E[X_n]$  and  $Var(X_n)$ .
- (iv) What is  $\pi$ , the probability of ultimate extinction, for this process?

[30 marks]

B5. (a) A gambler bets repeatedly on the toss of a coin which lands heads with probability  $\frac{1}{2}$  and tails with probability  $\frac{1}{2}$ . If the coin lands heads the gambler wins £1 and if tails, the gambler loses £1. If at any stage the gambler loses all of her money she receives an extra £1. Starting with £ $k$ , the gambler plays the game repeatedly until she reaches £ $N$  ( $N > k$ ). For  $k = 0, 1, \dots, N$ , let  $M_k$  be the number of bets until the conclusion of the game.

- (i) What is the value of  $E[M_N]$ ?
- (ii) Explain why  $E[M_0] = E[M_1]$ .
- (iii) For  $k = 1, 2, \dots, N - 1$ , show that

$$E[M_k] = 1 + \frac{1}{2}E[M_{k-1}] + \frac{1}{2}E[M_{k+1}].$$

- (iv) The solution of the difference equation obtained in (iii) is given by

$$E[M_k] = c_1 + c_2k - k^2$$

for some constants  $c_1$  and  $c_2$ . Using the boundary conditions obtained in (i) and (ii), find the values of the constants  $c_1$  and  $c_2$ , and prove that

$$E[M_k] = (N - k)(N + k - 1).$$

- (v) How does the expected number of bets for this problem compare with the expected number of bets in the standard gambler's ruin problem, i.e. the situation where the game will end if the gambler reaches either £0 or £ $N$ ?

- (b) Consider now the situation where a gambler bets repeatedly on the toss of a biased coin which lands heads with probability  $p$  and tails with probability  $q = 1 - p$ . If the coin lands heads, the gambler wins  $\mathcal{L}1$ , whereas, if the coin lands tails, the gambler loses  $\mathcal{L}1$ . Starting with  $\mathcal{L}k$ , the gambler plays the game repeatedly until he/she either goes bust or reaches  $\mathcal{L}N$ , where  $N > k$ . Let  $M_k$  be the number of tosses until the gambler either goes bankrupt or reaches  $\mathcal{L}N$ .

Suppose that the outcomes of the tosses *are not independent*. In particular, suppose that, for  $k = 2, 3, \dots$ , the outcome of the  $k$ -th toss is *identical* to the outcome of the first toss.

For  $k = 1, 2, \dots, N - 1$ , find  $E[M_k]$ .

[30 marks]

- B6. (a) Consider a homogeneous Markov Chain with states  $\{1, 2, 3, 4, 5, 6\}$  and one-step transition matrix given by

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

- (i) Produce a transition diagram for this Markov Chain and determine the communicating classes.
- (ii) Which states are transient and which are recurrent? Justify your answer.
- (b) The weather in a certain region can be characterised as sunny (S), cloudy (C) or rainy (R) on any particular day. The probability of any type of weather on one day depends only on the state of the weather on the previous day. For example, if it is sunny on one day then sun or clouds are equally likely on the next day with no possibility of rain.
- (i) Explain what the other day-to-day possibilities are if the changes in the weather are represented by the transition matrix

$$P = \begin{array}{c|ccc} & S & C & R \\ \hline S & \frac{1}{2} & \frac{1}{2} & 0 \\ C & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ R & 0 & \frac{1}{2} & \frac{1}{2} \end{array}.$$

- (ii) In the long-run, what percentage of days are sunny, cloudy and rainy?

[30 marks]